Transitionless quantum driving in open quantum systems

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OUTLINE

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Berry’s transitionless quantum driving

\[ \hat{H}_0(t) \left| \varphi_n(t) \right\rangle = E_n(t) \left| \varphi_n(t) \right\rangle \]

if the initial state of the system is an instantaneous eigenstate of a time dependent Hamiltonian \( H_0 \) it will remain the corresponding eigenstate at time \( t \) as long as \( H_0 \) varies slowly enough and there are no level crossing

\[ \hat{H}(t) = \hat{H}_0(t) + \hat{H}_1(t) \quad \hat{H}(t) \left| \varphi_n(t) \right\rangle = i \partial_t \left| \varphi_n(t) \right\rangle \]

transitionless quantum driving: add a new time dependent hamiltonian term \( H_1 \) so that the state becomes an exact solution regardless of the speed of change of the hamiltonian
a classical analog

a spin following a magnetic field whose direction changes in time
the rotating frame

\[ \hat{H}(t) = \sum_{i,j} |i\rangle \langle i| \hat{H}(t) |j\rangle \langle j| \]

\[ \hat{U}(t) = \sum_{i} |\varphi_i(t)\rangle \langle i| \]

\( \hat{U}(t) \) diagonalizes the hamiltonian

\[ \hat{U}^{-1}(t) \hat{H}(t) \hat{U}(t) \equiv \hat{H}_d(t) = \sum_{i} E_i(t) |i\rangle \langle i| \]
the time dependent Schoedinger eq.

\[ |\psi\rangle_d = \hat{U}^{-1}|\psi\rangle. \]  

\[ \hat{H}_d(t) + i\partial_t \hat{U}^{-1}(t)\hat{U}(t)|\psi\rangle_d = i\partial_t |\psi\rangle_d \]

\[ [\hat{H}_d(t) + \hat{H}_d'(t) + \hat{H}_{nd}(t)]|\psi\rangle_d = i\partial_t |\psi\rangle_d. \]

\[ \hat{H}_d'(t) = i \sum |i \rangle \langle i| \partial_t \hat{U}^{-1}(t)\hat{U}(t)|i\rangle \langle i| = i \sum \langle \phi_i | \phi_i \rangle |i\rangle \langle i|, \]

\[ \hat{H}_{nd}'(t) = i \sum_{i \neq j} |i \rangle \langle i| \partial_t \hat{U}^{-1}(t)\hat{U}(t)|j\rangle \langle j| = i \sum_{i \neq j} \langle \phi_i | \phi_j \rangle |i\rangle \langle j|, \]
transitionless quantum driving

Berry connection

\[ \hat{H}'_{d}(t) = i \sum_{i} \langle i | \partial_{t} \hat{U}^{-1}(t) \hat{U}(t) | i \rangle \langle i | = i \sum_{i} \langle \phi_{i} | \phi_{i} \rangle | i \rangle \langle i |, \]

non adiabatic term

\[ \hat{H}'_{nd}(t) = i \sum_{i \neq j} \langle i | \partial_{t} \hat{U}^{-1}(t) \hat{U}(t) | j \rangle \langle j | = i \sum_{i \neq j} \langle \phi_{i} | \phi_{j} \rangle | i \rangle \langle j |, \]

transitional quantum driving

\[ \hat{H}_{tqd}(t) = -\hat{U}(t) \hat{H}'_{nd}(t) \hat{U}^{-1}(t). \]
adiabatic approximation in open quantum systems

\[ \mathcal{L}[\rho] = -i[\hat{H}(t), \rho] + \frac{1}{2} \sum_{j=1}^{N} (2\hat{\Gamma}_j(t)\rho\hat{\Gamma}_j^\dagger(t) - \{\rho, \hat{\Gamma}_j^\dagger(t)\hat{\Gamma}_j(t)\}) \]

due to the coupling of the system with the environment, the energy-difference between neighbouring eigenvalues of the Hamiltonian no longer provides the natural time-scale with respect to which a time-dependent Hamiltonian could be considered to be slowly-varying.

adiabaticity of open systems is reached when the evolution of the state of a system occurs without mixing the various Jordan blocks into which L can be decomposed.

Sarandy Lidar PRA 71, 012331 (2005)
define a time independent basis in the D2-dimensional space of the density matrices. This could consist, for example, the three Pauli matrices and the identity matrix in the case of a single spin-1/2.

\[ B \equiv \{ \hat{\sigma}_i \} \quad i = \{1, \ldots , D^2 \}. \]

the density operator becomes a vector

\[ |\rho\rangle = (\rho_1, \rho_2, \ldots , \rho_{D^2})^\dagger, \]

the Lindblad operator becomes a super matrix

\[ L(t)|\rho\rangle = |\dot{\rho}\rangle. \]

\[ \rho_j = \text{Tr}[\hat{\sigma}_j^\dagger \rho] \quad L_{jk}(t) = \text{Tr}[\hat{\sigma}_j^\dagger (\mathcal{L}_t[\hat{\sigma}_k])]. \]
Although the supermatrix \( L(t) \) might be non-Hermitian, in which case it cannot be diagonalized in general, it is always possible to find a similarity transformation \( C(t) \) such that \( L(t) \) is written in the canonical Jordan form

\[
L_J(t) = C^{-1}(t)L(t)C(t) = \text{diag}[J_1(t), \ldots, J_N(t)],
\]

\[
C(t) = \sum_{\nu=1}^{N} \sum_{\mu_\nu=1}^{M_\nu} |D_{\nu,\mu_\nu}(t)\rangle \langle \langle \sigma_{\nu,\mu_\nu} |,
\]

\[
L(t)|D_{\nu,\mu_\nu}(t)\rangle \rangle = |D_{\nu,\mu_\nu-1}(t)\rangle \rangle + \lambda_\nu(t)|D_{\nu,\mu_\nu}(t)\rangle \rangle,
\]

\[
|D_{\nu,0}(t)\rangle \rangle \quad \text{represents the eigenvector of } L(t) \quad \text{corresponding to the eigenvalue } \lambda_\nu(t)
\]
transitionless open dynamics

formal analogy with the unitary case

\[(L_J + L'_J + L'_{nd})|\varrho\rangle_J = |\dot{\varrho}\rangle_J,\]

\[L'_{J} = \sum_{\nu} |\sigma_{\nu,\mu_{\nu}}\rangle\langle\langle\sigma_{\nu,\mu_{\nu}} | \dot{C}^{-1} C |\sigma_{\nu,\mu_{\nu}}\rangle\langle\langle\sigma_{\nu,\mu_{\nu}} |\]

\[L'_{nd} = \sum_{\nu \neq \nu'} |\sigma_{\nu,\mu_{\nu}}\rangle\langle\langle\sigma_{\nu,\mu_{\nu}} | \dot{C}^{-1} C |\sigma_{\nu',\mu_{\nu'}},\rangle\langle\langle\sigma_{\nu',\mu_{\nu'}}, |\]

transitionless quantum driving

\[L_{tqd} = -CL'_{nd}C^{-1}.\]

the driving term can be unitary (hamiltonian)
or non unitary (a quantum channel)

for one dimensional Jordan blocks

the off diagonal

matrix term of the correction term are

\[
\left\langle \left\langle \mathcal{D}_i(t) | \mathcal{D}_j(t) \right\rangle \right\rangle = \frac{\left\langle \left\langle \mathcal{D}_i(t) | \mathcal{L}(t) | \mathcal{D}_j(t) \right\rangle \right\rangle}{\lambda_j - \lambda_i}.
\]
rotating jump operators and unitary driving

\[ \mathcal{L}[\rho] = \sum_{k} \frac{\gamma_k}{2} \left[ 2 \hat{\Gamma}^k(t) \rho \hat{\Gamma}^k(t) - \{ \hat{\Gamma}^k(t) \hat{\Gamma}^k(t), \rho \} \right], \]

\[ \hat{\Gamma}^k(t) = \hat{U}^\dagger(t) \hat{\Gamma}_0^k \hat{U}(t) \]

in the rotating frame

\[ \ddot{\rho} = \hat{U}(t) \rho(t) \hat{U}^\dagger(t) \]

\[ \dot{\rho} = \sum_{k} \frac{\gamma_k}{2} \left[ 2 \hat{\Gamma}_0^k \dot{\rho} \hat{\Gamma}_0^k - \{ \hat{\Gamma}_0^k \hat{\Gamma}_0^k, \dot{\rho} \} \right] - i \left[ i \hat{U}(t) \hat{U}^\dagger(t), \tilde{\rho} \right] \]

the quantum unitary driving

\[ \hat{H}_{\text{td}}(t) = i \hat{U}(t) \hat{U}^\dagger(t). \]
example 1: single spin amplitude damping

\[ \mathcal{L}_{\text{ad}}[\rho] = \frac{\gamma}{2} \left[ 2\hat{\sigma}_n^- \rho \hat{\sigma}_n^+ - \{ \hat{\sigma}_n^- \hat{\sigma}_n^+, \rho \} \right] \]

\[ \hat{\sigma}_n^- = (\hat{\sigma}_n^+)^\dagger = |\downarrow\rangle_n \langle \uparrow| \]

amplitude damping along a time dependent direction

\[ n = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \]

precession around the z axis

\[ \hat{B} \equiv (\hat{I}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z). \]

\[ \hat{H}_{\text{tqd}}(t) = (n \times \dot{n}) \cdot \hat{\sigma}, \]

\[ \phi = \omega t \]

\[ \mathcal{L}_{\text{tqd}}[\rho] = -i \left[ \hat{H}_{\text{tqd}}(t), \rho \right] \]

\[ \hat{H}_{\text{tqd}}(t) = i \hat{U} \hat{U}^\dagger, \]
a two qubit example

\[
\Gamma_1 = \hat{U} \left( |0\rangle_1 \langle 1| \otimes \hat{I}_2 \right) \hat{U}^\dagger; \quad \Gamma_2 = \hat{U} \left( \hat{I}_1 \otimes |0\rangle_2 \langle 1| \right) \hat{U}^\dagger
\]

where U is a Hadamard gate followed by a C-NOT

the liouvillian has the following fixed point: \[ |\psi\rangle = \left(1/\sqrt{2}\right) (|00\rangle + |11\rangle) \]

let’s generalise by assuming U is an arbitrary single qubit rotation followed by a C-NOT. In this case the fixed point is \[ |\psi(t)\rangle = \left(\cos \theta(t) |00\rangle + \sin \theta(t) |11\rangle\right) \]

by rotating q one can drag the fixed point
a two qubit example

by varying slowly $\theta$ one can drag the fixed point

such dragging can be achieved exactly with no constrains on speed by adding the following coherent driving:

\[
H_{tqd} = i \hat{U} \hat{U}^\dagger = \begin{pmatrix}
0 & 0 & 0 & -i\dot{\theta} \\
0 & 0 & -i\dot{\theta} & 0 \\
0 & i\dot{\theta} & 0 & 0 \\
i\dot{\theta} & 0 & 0 & 0
\end{pmatrix}
\]

\[
H_{tqd} = -i\dot{\theta} \left( |00\rangle \langle 11| + |01\rangle \langle 10| \right) + \text{h.c.}
\]